# CALCULUS HANDOUT
## Optimization Problems

1. A rectangular sheet of paper is to contain 72 square inches of printed matter with 2-inch margins at top and bottom and 1-inch margins on each side. What dimensions for the sheet will use the least paper?

2. A publisher decides to print the pages of a large book with ½-inch margins on the top, bottom, and one side, and a 1-inch margin on the other side (to allow for the binding). The area of the entire page is to be 96 square inches. Find the dimensions of the page that will maximize the printed area of the page.

3. A manufacturer of Christmas tree ornaments knows that the total cost $C$ in dollars of making $x$ thousand ornaments of a certain kind is given by $C(x)=600+60x$ and that the corresponding sales revenue $R$ in dollars is given by $R(x)=300x-4x^2$. Find the number of ornaments that will maximize the manufacturer's profit.

4. A chemical manufacturer sells sulfuric acid in bulk at a price of \$100 per unit. If the daily total production cost in dollars for $x$ units is $C(x)=100{,}000+50x+0.0025x^2$ and if the daily production capacity is at most 7000 units, how many units of sulfuric acid must be manufactured and sold daily to maximize the profit? Would it benefit the manufacturer to expand the daily production capacity?

5. A firm determines that $x$ units of its product can be sold daily at $p$ dollars per unit, where $x = 1000 - p$. The cost of production $x$ units per day is $C(x)=3000+20x$.
   (a) Find the revenue function $R(x)$.
   (b) Find the profit function $P(x)$.
   (c) Assuming that the production capacity is at most 500 unites per day, determine how many units the company must produce and sell each day to maximize the profit.
   (d) Find the maximum profit.
   (e) What price per unit must be charged to obtain the maximum profit?

6. A closed rectangular container with a square base is to have a volume of 2250 $\text{in}^3$. The material for the top and bottom of the container will cost \$2 per $\text{in}^2$, and the material for the sides will cost \$3 per $\text{in}^2$. Find the dimensions of the container of least cost.

7. A cylindrical can, open at the top, is to hold 500 $\text{cm}^3$ of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.

8. Find the dimensions of a cylindrical closed can of the largest volume if its surface area is $32\pi$ square centimeters.

9. A commercial cattle ranch currently allows 20 steers per acre of grazing land; on the average its steers weigh 2000 lb at market. Estimates by the Agriculture Department indicate that the average market weight per steer will be reduced by 50 lb for each additional steer added per acre of grazing land. How many steers per acre should be allowed in order for the ranch to get the largest possible total market weight for its cattle?

10. A cable television company plans to begin operations in a small town. It foresees that about 600 people will subscribe to the service if the price per subscriber is \$5.00 per month. Experience shows that for each 5-cent increase in the individual subscription price per month, 4 of the original 600 people will decide not to subscribe. The cost to the company per month per subscription is estimated to be \$3.50.
    (a) What price per month per subscription will bring in the greatest total revenue to the company?
    (b) What price will bring in the greatest profit to the company?

11. A ferry transports tourists to the Middle Bass Island on Lake Erie during the summer months. The one-way fare is \$6.00 a person and 200 people ride the ferry each day. The owner estimates that for every \$0.50 the fare is raised, they will lose 10 customers. What should the fare be for the greatest income for the ferry owner?

12. A vacationer runs out of gas in a trailer park. He is at point A, directly 1 mile from a point $D$ on a paved road. He can reach a gas station at point $C$ by walking through the woods in a straight line from A to a point $B$ on the paved road at the rate of 3 miles per hour and then proceeding on the paved road at the rate of 5 miles per hour until he reaches $C$. If the distance from $D$ to $C$ is 4 miles along the paved road, how far should $B$ be from $D$ in order that he reach the gas station $C$ in the shortest time?

13. Points D & E are located 12 miles apart on a long, straight shoreline. Ship A is anchored 6 miles offshore from Point D while Ship B is anchored 18 miles offshore from point E. A boat from Ship A is to land a passenger on shore at point C, between D & E, and then proceed to Ship B. How far should C be located from D so that total distance traveled is minimized? Solve using CAS.

## ANSWERS

1. 8 in. x 16 in.
2. 12 in. x 8 in.
3. Produce 30,000 units
4. (a) 7000 units (b) Yes; The critical point was 10,000. They need to expand their capacity.
5. (a) $R(x) = 1000x - x^2$

   (b) $P(x) = 980x - x^2 - 3000$

   (c) 490 units
   (d) \$237,100
   (e) \$510
6. 15″ x 15″ x 10″
7. $r = \sqrt[3]{\dfrac{500}{\pi}}$ cm $\cong 5.42$ cm, $h = \sqrt[3]{\dfrac{500}{\pi}}$ cm $\cong 5.42$ cm
8. $h = \dfrac{8}{\sqrt{3}}$ cm, $r = \dfrac{4}{\sqrt{3}}$ cm
9. 30 steers per acre
10. (a) \$6.25 (b) \$8.00
11. \$8.00
12. $\dfrac{3}{4}$ mile from D
13. C is 3 miles from D. Total distance = 26.8 miles

# EXAMPLE OPTIMIZATION PROBLEMS

1. A farmer wishes to enclose a rectangular field with a fence and then divide the enclosed area into 3 sections by putting two fences down the middle. He only has 3000 feet of fencing on hand and does not want to buy more. What dimensions should he use to form the perimeter so that it encloses the largest possible area?

2. The farmer in problem #1 liked his three new cattle pens, so he has decided to build another pen of the same design. However, he wants to be sure to enclose 303,750 $ft^2$ of area. He will have to buy more fencing. If he uses fencing that costs \$1 per foot for the exterior fence and fencing that costs \$0.50 per foot for the interior fences, what dimensions should he use for the exterior fence to minimize his cost?

3. Smilin' Sam Simpson has decided to run for a seat in the Kansas Legislature. He plans to put up many campaign posters. The posters must contain 50 $in^2$ of printed campaign information with margins of 4 inches at the top and bottom and margins of 2 inches on the sides. Find the overall dimensions of the posters if he wants to minimize that amount of paper used.

4. A toy manufacturer knows that the total cost C in dollars of making $x$ thousands of toys is given by the equations $C(x)=600+3x$ and the corresponding sales revenue R in dollars is given by $R(x)=4x-0.0002x^2$. When production is begun, at least 1000 toys must be made. Production capacity is 10,000,000 toys. Find the optimum number of toys to produce and sell that will maximize the profit.

5. A gasoline station selling $x$ gallons of fuel per month has fixed costs of \$2500 and variable costs of \$0.90 per gallon. The price of the gasoline is determined by the function $p=\$1.50-0.00002x$ and the station's capacity allows no more than 20,000 gallons to be sold per month. Find the maximum profit.

6. An open-top box is to be made from a piece of cardboard 4 feet long and 2½ feet wide by cutting out squares of equal size at each corner and bending up the flaps. What should be the size of the squares to be removed in order to produce the maximum volume?

7. A closed box with a square base to ship technological equipment is to cost no more than \$756. The bottom costs \$5 per square foot, the top costs \$2 per square foot, and the sides cost \$3 per square foot. Find the dimensions that will maximize the volume.

8. A manufacturer makes aluminum cups in the form of right circular cylinders open at the top (no handle), having a volume of $16\pi$ $in^3$. If the cost of the material for the bottom is twice that for the sides, find the dimensions that will give the lowest cost.

9. An apple orchard owner estimates that if 24 trees per acre are planted, each tree will produce 600 apples per year (when mature). For every tree added per acre, each tree will produce 12 less apples. How many trees should be planted per acre to maximize production?

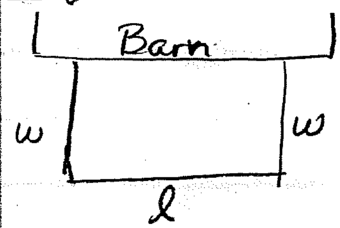
10. (a) A military courier is located on a desert 6 miles from a point P, which is the point on a long, straight road nearest to him. He is ordered to get to point Q, which is 3 miles down the road from point P. If he can travel 14 mph on the desert and 50 mph on the road, find the point R where he should reach the road in order to get to Q in the shortest possible time.

11. An oil drilling platform in the Gulf of Mexico is 9 km from point $A$, the nearest point on shore. A second oil drilling platform is 3 km from the nearest point $B$ on the shore. The distance from $A$ to $B$ is 5 kilometers. A supply depot is to be built at a point $C$ on the shore, between $A$ and $B$ in such a way that the sum of the distances from $C$ to the two platforms is a minimum. How far is it from $A$ to $C$? (Solve using your TI-89.)

# OPTIMIZATION

## Day 1

14a.




$A = lw$

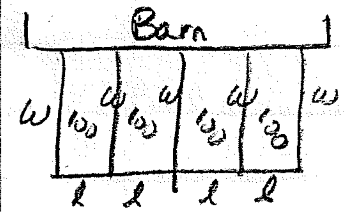
$l + 2w = 200$

Interval for $w$: $(0, 100)$

Interval for $l$: $(0, 200)$

Solution: 100 m x 50 m

14 b.




$F = 4l + 5w$

$wl = 100$

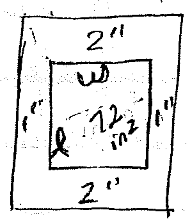
Interval for $w$: $(0, \infty)$

Interval for $l$: $(0, \infty)$

Solution: $5\sqrt{5}$ m x $4\sqrt{5}$ m

$\approx$ 11.18 m x 8.94 m

Handout #1



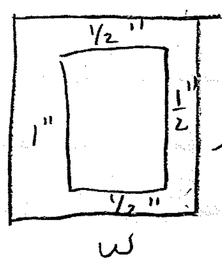

$A = (w+2)(l+4)$

$lw = 72$

Interval for $l$ or $w$: $(0, \infty)$

Solution: 8 in x 16 in.

Handout #2




$A = (w - 3/2)(l - 1)$

$lw = 96$

Interval for $l$ or $w$: $(0, \infty)$

Solution: 8 in X 12 in.

Handout #3 $P(x) = (300x - 4x^2) - (600 + 60x)$ Interval: $[0, \infty)$

Solution: Produce 30,000 units

Handout #4 Revenue = (price)(# of units) = $100x$ Interval: $[0, 7000]$

$P(x) = 100x - (100{,}000 + 50x + 0.0025x^2)$ Solution: (a) 7000 units

(b) Yes; The critical point was 10,000. They need to expand their capacity.

## Day 1 (cont.)

Handout #5 (a) Revenue = (price)(# of units) Since $x = 1000 - p$,

$= (p)(x)$ $\quad p = 1000 - x$

$= (1000 - x)(x)$

$R(x) = 1000x - x^2$

(b) $P(x) = (1000x - x^2) - (3000 + 20x)$

$P(x) = 980x - x^2 - 3000$

(c) Interval: $[0, 500]$ Solution: **490 units**

(d) $P(490) =$ **\$237,100**

(e) $p = 1000 - x$ $\quad p = 1000 - 490 =$ **\$510**

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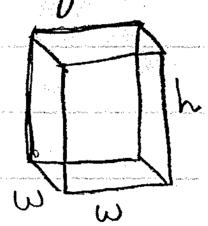
## Day 2

16.




$V = w^2 h$

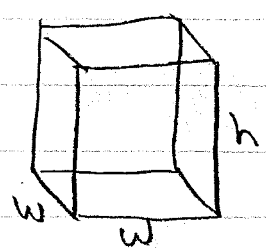
$w + w + h = 108$

Interval for w: $(0, 54)$

Interval for h: $(0, 108)$

Solution: 36" x 36" x 36"

17.

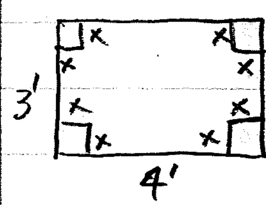



$C = 2 \cdot w^2 + \frac{1}{2}w^2 + 1.4wh$ Interval for w or h: $(0, \infty)$

$w^2 h = 16$

Solution: $\frac{4}{\sqrt[3]{5}}$ ft x $\frac{4}{\sqrt[3]{5}}$ ft x $\sqrt[3]{25}$ ft

32a.

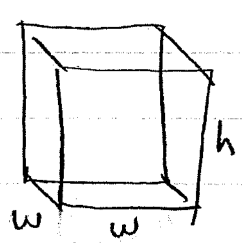



$V = (3 - 2x)(4 - 2x)x$ Interval: $(0, 1.5)$

Solution: $3.03 ft^3$

(Cut squares are 0.57 ft.)

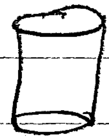
Handout #6.




$C = \$2(2w^2) + \$3(4wh)$ Interval: $(0, \infty)$

$w^2 h = 2250$

Solution: 15" x 15" x 10"

Handout #7 

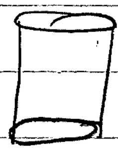
$A = \pi r^2 + 2\pi r h$

$\pi r^2 h = 500$

Interval: $(0, \infty)$

Solution: $r = \sqrt[3]{\frac{500}{\pi}}$ cm $h = \sqrt[3]{\frac{500}{\pi}}$ cm

$r = 5.42$ cm $h = 5.42$ cm

Handout #8 

$V = \pi r^2 h$

$2(\pi r^2) + 2\pi r h = 32\pi$

Interval for r: $(0, 4)$

Solution: $r = \frac{4}{\sqrt{3}}$ cm

$h = \frac{8}{\sqrt{3}}$ cm

Handout #9 $W = (20 + x)(2000 - 50x)$ Interval: $[0, 40]$

$x =$ # of steers added Solution: 30 steers per acre

Handout #10 (a) $R(x) = (5 + 0.05x)(600 - 4x)$ Interval: $[0, 150]$

$x =$ # of 5¢ increases Solution: \$6.25

(b) $P(x) = (5 + 0.05x)(600 - 4x) - \$3.50(600 - 4x)$

Interval: $[0, 150]$

Solution: \$8.00

Handout #11 $I = (6 + 0.5x)(200 - 10x)$ Interval: $[0, 20]$

$x =$ # of \$0.50 increases Solution: \$8.00

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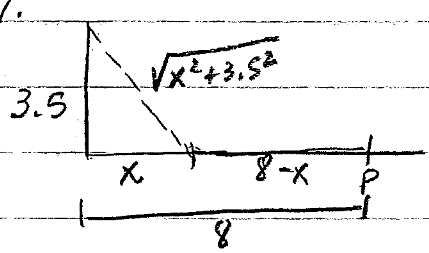
Day 3

37. 

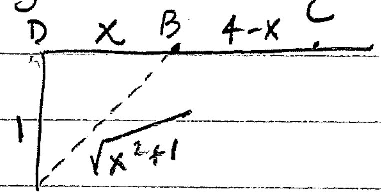

$C(x) = 2400\sqrt{x^2 + 3.5^2} + 1200(8 - x)$

Interval: $[0, 8]$

Solution: $\frac{7\sqrt{3}}{6}$ mi $\approx 2.02$ mi

Day 3 (cont.)

Handout #12

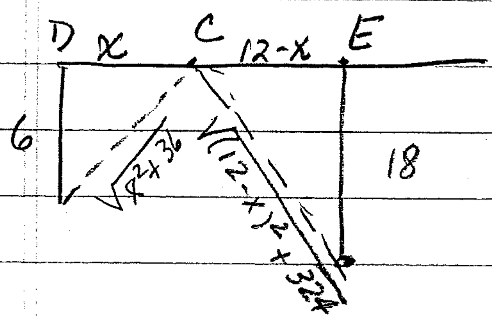



$$T = \frac{\sqrt{x^2+1}}{3} + \frac{4-x}{5}$$

Interval: [0, 4]

Solution: 3/4 mi from D

Handout #13




$$D(x) = \sqrt{x^2+36} + \sqrt{(12-x)^2+324}$$

Solution: 3 mi. from D

Distance: 26.8 mi

Interval: [0, 12]

| |
|---|
| Assignment: 8<br>complete problems |

Name____________________

**CALCULUS REVIEW**
**OPTIMIZATION**

1. Find the volume of the largest box that can be made from a piece of cardboard that has sides of 10 cm by cutting equal squares from the corners and turning up the sides.

2. A company estimates that if $x$ thousand units of some commodity are produced, then the total cost $C(x)$ and the total revenue $R(x)$ will be given by the equations $C(x)=\frac{x^3}{3}-4x^2+10x$ and $R(x)=10x-2x^2$. If the company can produce at most 5000 units, find the number of units that should be produced to yield the maximum profit.

3. Platypus owns some land for future expansion. In the meantime they rent the land as plots to gardeners. They find that when the rent is $250 per season, they are able to rent all 200 lots, but for every $10 increase in rent, 5 of the plots go unrented. What rent should Platypus charge to obtain the largest gross income?

4. A printed page must contain 60 $cm^2$ of printed material. There are to be margins of 5 cm on either side and margins of 3 cm on the top and bottom. What should the dimensions of the page be to minimize the amount of paper used?

5. To prevent animals from harming themselves on sharp pieces of scrap metal, Platypus Metalworks Corporation wants to fence off a rectangular scrap yard. Since the company is near the river, you decide to fence a rectangular plot along the river so that you need fence only three sides. The yard must have at least 480 square meters of area. However, the managers are concerned that the fence will be unsightly to passersby. You are told you must used decorative fencing along the front (parallel to the river). The decorative fencing costs \$25 per meter, while the standard fencing costs \$15 per meter. What dimensions will you use to enclose the required area yet minimize the cost to the company?

6. A man in a rowboat is 6 miles from point B on the shore. He wants to reach his home which is 8 miles down the shore from point B as quickly as possible. If he can walk 4 mph and row 2 mph, where should he land in order to reach his destination in the shortest possible time?

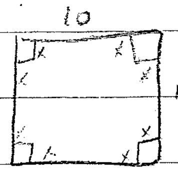
7. A cylindrical can with closed bottom and closed top is made from two kinds of material. The material used to make the bottom and top of the can costs \$.06 per square inch, and the material used to make the curved surface of the can costs \$.03 per square inch. The total cost of the can is \$1.44. Find the dimensions of the can the will maximize the volume.

8. An outdoor track is to be created in the shape shown and is to have a perimeter of 440 yards. Find the dimensions for the track that maximize the area of the rectangular portion of the field enclosed by the track.

9. A rectangular box is to be constructed containing 200 cu. inches. The base is to be made of material that costs \$0.43 per square inch while the material for the top and sides costs only \$0.29 per square inch. The base is to be 4 times as long as it is wide. The cost of joining all adjacent sides (except the top) is \$0.02 for each inch of edge, while the cost of hinging the top along the long side is \$1.15 per inch. What dimensions of the box will minimize the cost of the box? Round dimensions to two decimal places. CAS may be used, if desired.

10. Two houses are being built 150 feet apart. House A is 50 feet from the street while House B is 75 feet from the street. At what point on the edge of the street should a power pole be located to minimize the length of wire that must be run to the two houses? How much wire will be needed? (Solve using CAS.)

# Optimization Review Answers

1. 


$V = (10-2x)(10-2x)x$ $\qquad (0, 5)$

$V = (100 - 40x + 4x^2)x$ $\qquad \lim_{x \to 0} V = 0$

$V = 4x^3 - 40x^2 + 100x$ $\qquad \lim_{x \to 5} V = 0$

$V' = 12x^2 - 80x + 100 = 0$ $\qquad V(5/3) = \frac{20}{3} \cdot \frac{20}{3} \cdot \frac{5}{3} = \frac{2000}{27}$

$4(3x^2 - 20x + 25) = 0$

$4(3x-5)(x-5) = 0$

$x = 5/3, \cancel{5}$

**Volume is $\frac{2000}{27}$ $cm^3$**

2. $P(x) = 10x - 2x^2 - \left(\frac{x^3}{3} - 4x^2 + 10x\right)$

$P(x) = 2x^2 - x^3/3$ $\qquad [0, 5]$

$P'(x) = 4x - x^2 = 0$ $\qquad P(0) = 0$

$x(4 - x) = 0$ $\qquad P(4) = 10.67$

$x = 0 \quad x = 4$ $\qquad P(5) = 8.33$

**Produce 4000 units**

3. $I = (\#\text{ of items})(\text{price})$ $\qquad x = \#$ of \$10 increases $\qquad [0, 40]$

$I = (200 - 5x)(250 + 10x)$ $\qquad I(0) = 50,000$

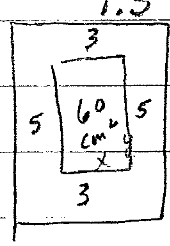
$I = 50,000 + 2000x - 1250x - 50x^2$ $\qquad I(40) = 0$

$I = 50,000 + 750x - 50x^2$ $\qquad I(7.5) = 52,812.50$

$I = 750 - 100x = 0$

$750 = 100x$ $\qquad$ Rent $= 250 + 10(7.5) =$ **\$325**

$7.5 = x$

4. 


$xy = 60 \qquad y = 60/x$

$A = (x+10)(y+6)$

$A = (x+10)\left(\frac{60}{x} + 6\right)$

$A = 60 + 6x + \frac{600}{x} + 60$

$A = 120 + 6x + \frac{600}{x}$

$A' = 6 - \frac{600}{x^2} = 0$

$6 = \frac{600}{x^2}$

$6x^2 = 600$

$x^2 = 100$

$x = \pm 10$

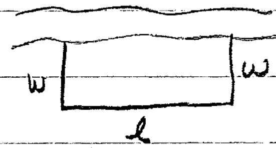
$(0, \infty)$

$\lim_{x \to 0} 120 + 6x + \frac{600}{x} = 120 + 0 + +\infty = +\infty$

$\lim_{x \to \infty} 120 + 6x + \frac{600}{x} = 120 + \infty + 0 = +\infty$

$A(10) = 240$

**Dimensions: 20 cm x 12 cm**

5. 


$lw = 480 \quad l = \frac{480}{w}$

$C = 25l + 15(2w)$

$C = 25\left(\frac{480}{w}\right) + 30w$

$C = \frac{12{,}000}{w} + 30w$

$C' = -\frac{12000}{w^2} + 30 = 0$

$30 = \frac{12000}{w^2}$

$30w^2 = 12000$

$w^2 = 400$

$w = \pm 20$

$(0, \infty)$

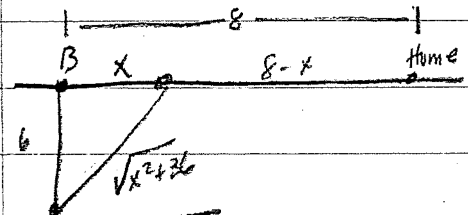
$\lim_{w \to 0} \frac{12000}{w} + 30w = +\infty$

$\lim_{w \to \infty} \frac{12000}{w} + 30w = +\infty$

20 must be the min

$l = \frac{480}{20} = 24$

**24 m x 20 m**

6. 


$T = \frac{\sqrt{x^2+36}}{2} + \frac{8-x}{4}$

$T = \frac{1}{2}(x^2+36)^{1/2} + 2 - \frac{1}{4}x$

$T' = \frac{1}{4}(x^2+36)^{-1/2} \cdot 2x - \frac{1}{4}$

$T' = \frac{x}{2\sqrt{x^2+36}} = \frac{1}{4}$

$(4x)^2 = \left(2\sqrt{x^2+36}\right)^2$

$16x^2 = 4(x^2+36)$

$16x^2 = 4x^2 + 144$

$12x^2 = 144$

$x^2 = 12$

$x = \pm 2\sqrt{3}$

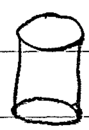
$[0, 8]$

$T(0) = 5$

$T(2\sqrt{3}) = 4.598$ *

$T(8) = 5$

**$2\sqrt{3}$ miles from B OR 3.46 miles from B**

7. 

$.06(2\pi r^2) + .03(2\pi rh) = 1.44$

$V = \pi r^2 h$

$.12\pi r^2 + .06\pi rh = 1.44$

Look here to find interval

$\frac{.06\pi rh}{.06\pi r} = \frac{1.44}{.06\pi r} - \frac{.12\pi r^2}{.06\pi r}$

$h = \frac{24}{\pi r} - 2r$

$V = \pi r^2\left(\frac{24}{\pi r} - 2r\right)$

$V = 24r - 2\pi r^3$

$V' = 24 - 6\pi r^2$

$24 = 6\pi r^2$

$\sqrt{\frac{4}{\pi}} = \sqrt{r^2}$

$1.13 \approx \sqrt{\frac{4}{\pi}} = r$

$.12\pi r^2 + 0 = 1.44$

$r^2 = 3.82$

$r = 1.95$

$(0, 1.95)$

$\lim_{r \to 0} 24r - 2\pi r^3 = 0$

$\lim_{r \to 1.95} 24r - 2\pi r^3 = 0$

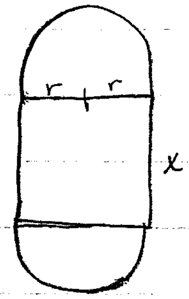
$V\left(\sqrt{\frac{4}{\pi}}\right) = 18.05$

$h = \frac{24}{\pi\left(\sqrt{\frac{4}{\pi}}\right)} - 2\sqrt{\frac{4}{\pi}} = 4.51$

$r \approx 1.13$ in.

$h \approx 4.51$ in.

8.




$2x + 2\pi r = 440 \rightarrow 2x = 440 - 2\pi r \rightarrow x = 220 - \pi r$

$A = x \cdot 2r$

$A = (220 - \pi r)2r$

$A = 440r - 2\pi r^2$

$A' = 440 - 4\pi r = 0$

$440 = 4\pi r$

$\frac{110}{\pi} = r$

$\left(0, \frac{220}{\pi}\right)$ or $(0, 70.03)$

$\lim_{r \to 0} 440r - 2\pi r^2 = 0$

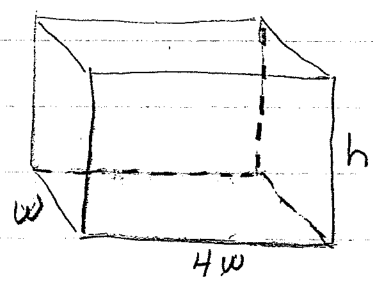
$\lim_{r \to \frac{220}{\pi}} 440r - 2\pi r^2 = 0$

$A\left(\frac{110}{\pi}\right) = 7703.1$ *

$x = 220 - \pi\left(\frac{110}{\pi}\right) = 110$

**$x = 110$ yds $\quad r = \frac{110}{\pi}$ yds**

9.




$4w^2 h = 200 \rightarrow h = \frac{200}{4w^2} = \frac{50}{w^2}$

$C = .43(4w^2) + .29(4w^2) + .29[2(4wh) + 2(wh)]$

$+ .02[2(4w) + 2w + 4h] + 1.15(4w)$

$C = 2.88w^2 + 2.9wh + 4.80w + 0.08h$

$C = 2.88w^2 + 2.9w\left(\frac{50}{w^2}\right) + 4.80w + 0.08\left(\frac{50}{w^2}\right)$

$C = 2.88w^2 + \frac{145}{w} + 4.80w + \frac{4}{w^2}$

$C' = \left[5.76w - \frac{145}{w^2} + 4.80 - \frac{8}{w^3}\right]$

$5.76w^4 - 145w + 4.80w^3 - 8 = 0$

$w = 2.7$ $\qquad h = \frac{50}{2.7^2} = 6.86$

$(0, \infty)$

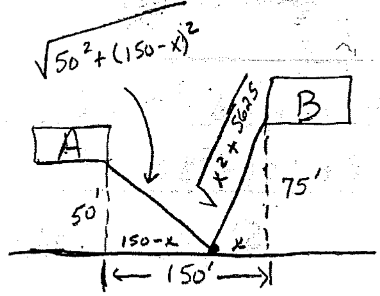
$\lim_{w \to 0} C = +\infty$

$\lim_{w \to \infty} C = +\infty$

$C(2.7) = 88.21$

**10.8 in x 2.7 in x 6.86 in**

10.




$d = \sqrt{50^2 + (150-x)^2} + \sqrt{x^2 + 5625}$

$d' = \frac{x - 150}{\sqrt{x^2 - 300x + 25000}} + \frac{x}{\sqrt{x^2 + 5625}}$

$x = 90$

$[0, 150]$

$d(0) = 233.114$

$d(150) = 217.705$

$d(90) = 195.256$

The pole should be located 90 feet along the edge of the street from House B or 60 feet from House A. The total wire needed is approximately 195.3 ft.